

MINIMUM-WEIGHT CODEWORDS OF THE HERMITIAN CODES ARE SUPPORTED ON COMPLETE INTERSECTIONS

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ABSTRACT. Let \mathcal{H} be the Hermitian curve defined over a finite field \mathbb{F}_{q^2} . Aim of the present paper is to complete the geometrical characterization of the supports of the minimum-weight codewords of the algebraic-geometry codes over \mathcal{H} , started in [13]. In that paper we considered the codes with distance $d \geq q^2 - q$ and proved that the supports of minimum-weight codewords are cut on \mathcal{H} by curves varying in a single family. Here we deal with the remaining Hermitian codes and show that the above result holds true only for some of the codes with lower distance. In fact, the minimum-weight codewords of codes with distance $d < q$ are supported on the complete intersection of two curves none of which is \mathcal{H} . Moreover, there are some very special code with distance $q \leq d \leq q^2 - q$ whose minimum weight codewords are of two different types: some of them are supported on complete intersections of \mathcal{H} and curves in a given family, some are supported on complete intersections of two curves none of which is \mathcal{H} .

1. INTRODUCTION

Let \mathbb{F}_{q^2} be the finite field with q^2 elements, q a power of a prime. The *Hermitian curve* \mathcal{H} is the affine, plane curve over \mathbb{F}_{q^2} defined by the polynomial $x^{q+1} = y^q + y$. It is a smooth curve of genus $g = \frac{q^2 - q}{2}$ with only one point at infinity. The curve \mathcal{H} is the most studied example of *maximal curve*, that is with the maximum number of \mathbb{F}_{q^2} -points allowed by the Hasse-Weil bound [16]. For this reason it is well suited for the construction of algebraic-geometric codes.

After Stichtenoth's work [17] several authors give significant contribution on this topic, we quote for instance [6, 7, 18] regarding a generic description of Hermitian codes and their distance, [8, 9, 10, 15] about the decoding of the Hermitian codes, [1, 3, 4, 11, 12] on small weight codewords and [2, 5, 14, 19] regarding generalized Hamming weights.

In this work we continue the geometrical description of the support of the minimum weight codewords of the Hermitian codes started in [13]. We also refer to that paper for a more detailed historical introduction and bibliography.

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Our main achievement is the following result.

Theorem A. *For any Hermitian code, the support of every minimum-weight codeword is a complete intersection of two plane curves.*

Making reference to the Høholdt, van Lint and Pellikaan work [7], every Hermitian code can be identified by an integer m , so we denote it as C_m , and can be classified in *four phases* depending on m (see Remark 2.9). A proof for Theorem A applied to III and IV phase is provided in [13]. In this paper we provide a proof of the same theorem applied to I and II phase.

A leading role in this work is played by the numerical semi-group $\Lambda = \langle q, q + 1 \rangle$. The discussion about the I and II phase turned out to be more difficult with respect to the one about III and IV, because of the presence of *gaps* for Λ among the values of m and among the distances $d(m)$ of C_m . Indeed, for each Hermitian code of III and IV phase, the supports of minimum weight codewords are complete intersection divisors cut on \mathcal{H} by a single family of curves, while in the I and II phase there are some special values of m , depending on distribution of the gaps, such that there are two different type of supports. Part of the minimum weight codewords of these special codes are supported on divisors of the previous shape, and the remain ones are supported on complete intersection of two curves of which neither is the Hermitian curve.

Note that there is a partial overlap with known results as far as it concerns the geometric description of minimum weight codewords. In particular, Marcolla, Pellegrini and Sala [12] find a geometric description for the codes in the I phase, while Ballico and Ravagnani [1] find it for some codes of the II phase. In Section 6 we analyze in detail similarities and differences between these two works and the present one.

Here we only underline that the methodologies we introduce allow us to describe features of the codes C_m , for a large range of values m , by a single formula and prove them with short proofs, avoiding case by case arguments.

We intend to further exploit the potentiality of these methods to also analyze the support of small weight codewords. Some results presented in this paper, as for instance those in Section 4, give information on this more general case. We are confident that, from this strong geometric characterization, also the explicit computation of the weight distribution will follow.

The paper is organized as follows:

- In Section 2 we recall some basic definitions and we prove some preliminary results about Hermitian codes C_m .

- In Section 3 we consider the codes in the I phase, computing their distance and providing the geometric description of their minimum-weight codewords, according to the results obtained in [12].
- In Section 4 we state and prove Theorem 4.1, which gives information about the small-weight codewords of C_m in the II phase, and in Theorem 4.2 we prove the distance of these codes.
- In Section 5 we focus on the \mathbb{F}_{q^2} -divisors that are the support of minimum weight codewords of codes C_m in the II phase, proving the main result of the present work, that is, *the support of every minimum-weight codeword of these codes is a complete intersection of two plane curves* (Theorem 5.2 and Theorem 5.4).
- Finally, in Section 6 we compare our results with those in [12, 1] and in Section 7 we draw the conclusions.

2. GENERALITIES AND PRELIMINARY RESULTS

2.1. **General notations.** In the paper we fix the following notations:

- (i): q is a power of a prime integer p and \mathbb{F}_{q^2} will denote the finite field with q^2 .
- (ii): Λ is the semi-group generated by q and $q + 1$. For every integer in Λ we always consider its unique writing $aq + b(q + 1)$ with $a \leq q$. Every positive integer that does not belong to Λ is called a *gap* of the semi-group;
- (iii): m is an integer number $0 \leq m \leq q^3 + q^2 - q - 2$, but we will focus mainly on the values $q^2 - 1 \leq m \leq 2q^2 - 2q - 3$.
- (iv): $\delta(m) := m - q^2 - q - 2$.

2.2. **The Hermitian curve.** Let \mathbb{F}_{q^2} be the finite field with q^2 elements, where q is a power of a prime and let K be its algebraic closure. For any ideal I in the polynomial ring $A := \mathbb{F}_{q^2}[x, y]$ we denote by $\mathcal{V}(I)$ the corresponding variety in \mathbb{A}_K^2 . If $g_1, \dots, g_s \in \mathbb{F}_{q^2}[x, y]$, we denote by $\langle g_1, \dots, g_s \rangle$ the ideal they generate.

The *Hermitian curve* \mathcal{H} is the curve in the affine plane \mathbb{A}_K^2 defined by the polynomial $H := x^{q+1} - y^q - y$. We will denote by $I_{\mathcal{H}}$ its defining ideal $\langle H \rangle$ in A and by $A_{\mathcal{H}}$ the coordinate ring $A/I_{\mathcal{H}}$ of \mathcal{H} .

The curve \mathcal{H} has genus $g := \frac{q(q-1)}{2}$ and $n := q^3$ closed points with coordinates in \mathbb{F}_{q^2} (\mathbb{F}_{q^2} -points for short), that we will always denote by P_1, \dots, P_n . We will denote by E the zero-dimensional scheme of degree n composed by all the \mathbb{F}_{q^2} -points of \mathcal{H} . For every point P_i its coordinates (x_i, y_i) satisfy the *norm-trace* condition $N(x_i) = \text{Tr}(y_i)$.

The projective closure $\overline{\mathcal{H}}$ of \mathcal{H} in \mathbb{P}_K^2 has only one, simple, point $P_{\infty} = [0 : 0 : 1]$ in $\overline{\mathcal{H}} \setminus \mathcal{H}$, so that $\overline{\mathcal{H}}$ has $q^3 + 1$ \mathbb{F}_{q^2} -points [16].

Definition 2.1. A \mathbb{F}_{q^2} -divisor over the Hermitian curve is a divisor $D = \sum_{i=1}^{\delta} Q_i$ where the Q_i 's are pairwise distinct \mathbb{F}_{q^2} -points of \mathcal{H} . We will denote by $|D|$ the degree δ of D . We can also write $D = \{Q_1, \dots, Q_{\delta}\}$; in particular, $E = \{P_1, \dots, P_n\}$ and D is a \mathbb{F}_{q^2} -divisor on \mathcal{H} if and only if $D \subseteq E$.

We denote by I_D the ideal generated by all polynomials in A vanishing on D and by A_D the quotient ring A/I_D .

Observe that in the above notations, we have $A_E = A/\langle H, x^{q^2} - x, y^{q^2} - y \rangle$; moreover, D is a \mathbb{F}_{q^2} -divisor on \mathcal{H} if and only if A_D is a quotient of A_E .

2.3. A quick sketch on the Hermitian codes as affine-variety codes. A linear code C over \mathbb{F}_{q^2} is a linear subspace of $\mathbb{F}_{q^2}^n$ for a suitable n , called length of C . The dual code C^{\perp} of C is formed by all vectors \mathbf{v} such that $G\mathbf{v}^T = 0$, where G is the generator matrix of C . Every generator matrix of C^{\perp} is called a *parity-check matrix* of the code C . The *weight* of a codeword $\mathbf{c} = (c_1, \dots, c_n) \in C$ is the number of c_i that are different from 0 and its *support* $\text{Supp}(\mathbf{c})$ is the set of indexes corresponding to the non-zero entries. The *distance* d of a linear code C is the minimum weight of its non-zero codewords.

We now briefly recall the definition the Hermitian code as an affine-variety code over the Hermitian curve \mathcal{H} . Let us consider *evaluation map*

$$(2.1) \quad \begin{aligned} \phi_E : A_E &\longrightarrow (\mathbb{F}_{q^2})^n \\ f &\longmapsto (f(P_1), \dots, f(P_n)). \end{aligned}$$

For every given linear subspace V of A_E and ordered set of generators $\langle f_1, \dots, f_s \rangle$ of V we can define the code $C(V) := \phi_E(V)$, whose *generator matrix* is the $s \times n$ matrix with entries $f_i(P_j)$, and its dual code $C(V)^{\perp}$. Note that the dual code depends on V , but does not depend on the chosen set of generators.

The Hermitian codes we consider here are dual codes $C(V)^{\perp}$ for the special linear subspaces V defined in the following way.

Definition 2.2. Let us fix the weight vector $w := [q, q+1]$ and associate to every monomial $x^r y^s$ the w -degree $w(x^r y^s) = rq + s(q+1)$. For every positive integer m , we denote by V_m the subspace of A_E generated by the (classes of) monomials of w -degree less than $m+1$. We will denote by C_m and call *Hermitian code* the dual code $C(V_m)^{\perp}$.

Therefore, the length of an Hermitian code is the number n of \mathbb{F}_{q^2} -points in \mathcal{H} and the entries of a codeword \mathbf{c} are labeled after these points, so that we can identify the support of \mathbf{c} with a divisor $D \subset E$ of the points corresponding to the non-zero entries of \mathbf{c} .

By definition, for every $m < m'$ we have $C_m \supseteq C_{m'}$.

Notation 2.3. For every Hermitian code C_m we will denote by \mathcal{D}_m the set of divisors on \mathcal{H} that are support of codewords of C_m ; moreover, we will denote by \mathcal{M}_m the set of those that are the support of minimum-weight codewords of C_m .

Usually the classes in A_E of monomials with w -degree less than $m+1$ are not linearly independent and, in order to obtain a parity-check matrix with maximal rank, it is convenient to select a suitable subset which form a basis for V_m . To choose this basis we fix the degrevlex term order with $y > x$, denoted by \prec , and consider the set of monomials

$$\mathcal{B} := \{x^r y^s \mid 1 \leq r \leq q, s \leq q^2 - q - 1\} \cup \{y^s \mid q^2 - q \leq s \leq q^2 - 1\}.$$

Their classes are linearly independent in A_E since they are the elements of the sous-escalier $\mathcal{N}(I_E)$ of the initial ideal of I_E w.r.t. \prec ; indeed $\text{In}_{\prec}(I_E) = \langle x^{q+1}, xy^{q^2-q}, y^{q^2} \rangle$.

With this choice, the set of monomials $\mathcal{B}_m := \mathcal{B} \cap V_m$ is a basis of V_m for every $m \geq 0$.

Remark 2.4. For every $m \geq 0$ the Hermitian code C_m is the code in $\mathbb{F}_{q^2}^n$ with parity-check matrix $\phi_E(\mathcal{B}_m)$ where ϕ_E is the evaluation map (2.1) at the points of E .

For every divisor D on \mathcal{H} we will denote by $V_{m,D}$ the image of V_m in A_D .

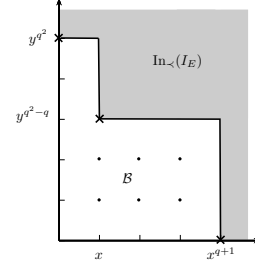
Remark 2.5. To find the distance of an Hermitian code, we will use a similar approach to that of [7]. The main difference concerns the choice of the term order used to find a basis of V . In fact, in [7] the term order is the weighed term $<_w$ associated to the weight vector $w = [q, q+1]$ (and Lex with $y > x$ as a “tie-breaker”). The basis given by our term order proves to be well suited to find geometrical description of minimum weight codewords.

In order to highlight similarities and differences with [7] we develop along all the paper the same example that we can also find in [7].

2.4. The gaps of Λ and the Hermitian codes. The semi-group Λ is related to many features of the Hermitian curve; for instance the number $q^2 - q - 1$ of gaps in Λ is equal to the geometric genus g of \mathcal{H} . The semigroup Λ and its gaps play a central role also in the study of the Hermitian codes. In the following remark we collect some features of the gaps and some interesting relation that concern the subset $\Lambda_{\mathcal{B}}$ of Λ of the w -weights of the monomial in \mathcal{B} .

Remark 2.6. (1) There are $q - 1$ “segments” of gaps of decreasing length; more precisely, for every $h = 1, \dots, q - 1$ there are the following $(q - h)$ gaps:

$$(2.2) \quad (h - 1)(q + 1) + 1, \quad \dots \quad (h - 1)(q + 1) + t, \quad \dots \quad (h - 1)(q + 1) + (q - h)$$



with $1 \leq t \leq q - h$. The last integer before such a sequence is $(h - 1)(q + 1) = w(y^{h-1})$ and the first integer after the sequence is $hq = w(x^h)$. Therefore, for every $h = 1, \dots, q - 2$, between to segments of gaps we find the following segment of element of Λ :

$$(2.3) \quad hq = w(x^h), \quad \dots \quad (h - i)q + i(q + 1) = w(x^{h-i}y^i), \quad \dots \quad h(q + 1) = w(y^h).$$

- (2) There is a 1 – 1 correspondence between Hermitian codes and elements in $\Lambda_{\mathcal{B}}$. In fact, if $m + 1 \notin \Lambda_{\mathcal{B}}$, then $\mathcal{B}_m = \mathcal{B}_{m+1}$, so $C_m = C_{m+1}$, while if $m + 1 \in \Lambda_{\mathcal{B}}$ the codes C_m and C_{m+1} are different. For this reason in the following **we will always label an Hermitian code by an integer m such that $m + 1 \in \Lambda_{\mathcal{B}}$** .
- (3) The w -weights of the monomials in \mathcal{B} are pairwise different, so that there is a 1 – 1 correspondence between \mathcal{B} and $\Lambda_{\mathcal{B}}$. Moreover, the monomials in \mathcal{B} are ordered by \prec in the same way as by the weighed term order with weight-vector w , namely they are ordered as the corresponding elements in $\Lambda_{\mathcal{B}}$ are ordered by $<$.
- (4) Due to the previous items, for every code C_m we can find in \mathcal{B} a unique monomial whose w -weight is $m + 1$.

2.5. The semi-group Λ and the distance of a Hermitian code C_m . In the previous paper [13] the authors prove that the distance $d(m)$ of the Hermitian codes C_m with $m \geq 2q^2 - 2q - 2$ is given by the integer $\delta(m) := m - (q - 2)(q + 1)$. This formula agrees with that proved by [7, 17, 18], taking in account our convention on the choice of the integer m labeling the code C_m given in Remark 2.6 (2).

Easy examples show that the formula for the distance $d(m) = \delta(m)$ is not correct for some of the Hermitian codes C_m in the range $q^2 - q - 1 \leq m \leq 2q^2 - 2q - 3$ considered in the present paper.

Example 2.7. For $q = 4$ let us consider the range $10 \leq m \leq 21$, containing the first integers m such that $\delta(m)$ is not negative. As already observed, $C_{10} = C_{11}$ since $10 + 1 \notin \Lambda_{\mathcal{B}}$; hence $d(10) = d(11)$, while $\delta(10) \neq \delta(11)$.

Moreover, by a direct computation (or using the known formula) we see that the distance of the Hermitian code C_{11} is $d(11) = 4$, while $\delta(10) = 0$ and $\delta(11) = 1$.

More generally, for m in the above range, we have $d(m) = \delta(m)$ if $m = 14, 15, 18, 19, 20$ and $d(m) \neq \delta(m)$ if $m = 10, 11, 12, 13, 16, 17, 21$. We underline that this second list contains precisely all the integers in the considered range such that either $m + 1 \notin \Lambda_{\mathcal{B}}$ ($m = 10$) or $\delta(m) \notin \Lambda_{\mathcal{B}}$. We will examine this case again in Example 2.10.

We now introduce a new function of m , that agrees with $\delta(m)$ for every $m \geq 2q^2 - 2q - 2$, so that it coincides with the distance $d(m)$ of C_m .

Notation 2.8. In the following we will denote by \tilde{m} the minimum integer $m' \geq m$ such that both $m' + 1 \in \Lambda_{\mathcal{B}}$ and $\delta(m') \in \Lambda$; moreover $\tilde{\delta}_m$ will denote the integer $\delta(\tilde{m}) = \tilde{m} - (q + 1)(q - 2)$.

Note that with our convention about the integer that labels a code C_m , there always exists a Hermitian code labeled after \tilde{m} and that $\tilde{\delta}_m$ is an element in $\Lambda_{\mathcal{B}}$.

Remark 2.9. In [7], Høholdt, van Lint and Pellikaan divide the Hermitian codes in four *phases*. There is a strict relation between these phases and the function \tilde{m} . In this paper we adopt their classification with minor changes.

- I phase::** $m \leq q^2 - 2$. For some of the integers m in this range, $m + 1$ is a gap and we do not label a code after them. Moreover, also for some of the remaining ones, $\delta(m)$ is a gap.
- II phase::** $q^2 - 1 \leq m \leq 2q^2 - 2q - 3$. All the integers $m + 1$ are in $\Lambda_{\mathcal{B}}$, but some of the $\delta(m)$ are gaps; then $m = \tilde{m}$ if and only if $\delta(m) \in \Lambda$.
- III phase::** $2q^2 - 2q - 2 \leq m \leq q^3 - 2$. Both $m + 1 \in \Lambda_{\mathcal{B}}$ and $\delta(m) \in \Lambda$, so that we always have $m = \tilde{m}$.
- IV phase::** $q^3 - 1 \leq m \leq q^3 + q^2 - q - 2$. Neither m , nor $\delta(m)$ are gaps, but for some m we have $m + 1 \notin \Lambda_{\mathcal{B}}$; then $m = \tilde{m}$ if and only if $m + 1 \in \Lambda_{\mathcal{B}}$.

The range we are mainly interested in the present paper corresponds to the I and especially the II phase. For m in this range we have $\delta(m) \leq q^2 - q - 1$; then some of them are negative and we also find among them all the gaps of Λ . Therefore, in correspondence with the gaps for $\delta(m)$ there are “segments” of values of m that share a same \tilde{m} . More precisely, looking at (2.2) and (2.3), we find a segment of integers m of this type for each $h = 1, \dots, q - 1$

$$(2.4) \quad (q+h-3)(q+1)+1, \quad \dots \quad (q+h-3)(q+1)+t, \quad \dots \quad (q+h-3)(q+1)+(q-h).$$

where $1 \leq t \leq q - h$. Then \tilde{m} is for all of them the first integer after the segment, namely:

$$(2.5) \quad \begin{cases} \tilde{m} = hq + (q-2)(q+1) = w(x^h y^{q-2}) \\ \tilde{m} + 1 = w(x^{h-1} y^{q-1}) \\ \tilde{\delta}_m = w(x^h). \end{cases}$$

In the following we fix an integer m such that $m \leq 2q^2 - 2q - 3$.

Example 2.10. Let us consider again the case of Example 2.7: $q = 4$ and $11 \leq m \leq 22$. In the following table we summarize the behavior of the three integers m , \tilde{m} and $\tilde{\delta}_m$; in black the segments of decreasing length 3,2,1 of the integers $\delta(m)$ that are gaps and the corresponding $m \neq \tilde{m}$.

m	11	12	13	14	15	16	17	18	19	20	21	22
\tilde{m}	14	14	14	14	15	18	18	18	19	20	22	22
$\delta(m)$	1	2	3	$4 = w(x)$	5	6	7	$8 = w(x^2)$	9	10	11	$12 = w(x^3)$
$\tilde{\delta}_m$	4	4	4	4	5	8	8	8	9	10	12	12

Note that for $11 \leq m \leq 14$ we are in the I phase, while the following values $m \leq 21$ correspond to the II phase and C_{22} is the first code in the III phase, already considered in [13].

In order to find the distance of the code C_m we study the degree of some \mathbb{F}_{q^2} -divisors D on the Hermitian curve. The following two results are crucial fact in our construction.

Proposition 2.11. *Let $F \in \mathbb{F}_{q^2}[x, y]$ be a polynomial such that $\partial_x(F) \leq q$ and let \mathcal{X} be the curve given by $F = 0$. If $\text{LM}_{\prec}(F) = x^r y^s$, then*

$$(2.6) \quad \text{In}_{\prec}(\langle H, F \rangle) = \langle x^{q+1}, x^r y^s, y^{s+q} \rangle$$

Moreover, the degree of the divisor D cut on \mathcal{H} by \mathcal{X} is $w(x^r y^s) = rq + s(q+1)$. More generally, if D is a divisor over \mathcal{H} and $x^r y^s$ is any monomial in $\text{In}_{\prec}(I_D)$ with $r \leq q$, then $|D| \leq rq + s(q+1)$.

Proof. See Proposition 2.9 of [13] and Corollary 2.10 of [13]. \square

Proposition 2.12. *Let D be a \mathbb{F}_{q^2} -divisor on the Hermitian curve. Then the following are equivalent*

- (i) $\exists \mathbf{c} \in C_m$ with $\mathbf{c} \neq 0$ such that $\text{Supp}(\mathbf{c}) \subseteq D$;
- (ii) $\dim(V_{m,D}) \leq |D|$ as \mathbb{F}_{q^2} vector space;
- (iii) $\exists x^u y^v \in \mathcal{N}(\text{In}_{\prec}(I_D))$ such that $m+1 \leq w(x^u y^v) \leq m+q+1$;

Proof. See Proposition 2.8 of [13]. \square

Corollary 2.13. *Let us consider two Hermitian code C_m and $C_{m'}$ with $m < m'$. Then*

$$\mathcal{D}_m \supseteq \mathcal{D}_{m'} \quad \text{and} \quad d(m) \leq d(m').$$

If, in particular, $d(m) = d(m')$, then $\mathcal{M}_m \supseteq \mathcal{M}_{m'}$.

Lemma 2.14. *Let D be a divisor over the Hermitian curve and let κ be any non-negative integer. Then*

$$y^{\kappa+q} \in \mathcal{N}(\text{In}(I_D)) \implies x^q y^{\kappa} \in \mathcal{N}(\text{In}(I_D)).$$

Proof. We prove the equivalent statement $x^q y^{\kappa} \in \text{In}(I_D) \implies y^{\kappa+q} \in \text{In}(I_D)$.

Let F be a monic polynomial in I_D such that $\text{In}(F) = x^q y^{\kappa}$; we can write it as $F = x^q y^{\kappa} + x^{q+1}G + M$ with $G = \sum_{i=1}^{\kappa} a_i x^{i-1} y^{\kappa-i}$ and $\partial M < q + \kappa$. Then the polynomial $S := xF - y^{\kappa}H - xGH$ belongs to I_D . We prove that its leading monomial is $y^{\kappa+q}$. By a straightforward computation we see that $S = y^{\kappa+q} + x(M + y^q G + yG)$ is a polynomial of degree $q + \kappa$, hence $y^{\kappa+q}$ is its maximum degree with respect to the term order \prec . \square

3. DISTANCE AND MINIMUM-WEIGHT CODEWORDS OF CODES IN THE I PHASE

Now we focus on the codes C_m with $m \leq q^2 - 2$, such that $m + 1$ belongs to a sequence

$$hq = w(x^h), \quad \dots \quad (h-i)q + i(q+1) = w(x^{h-i}y^i) \quad \dots \quad h(q+1) = w(y^h)$$

for some $1 \leq h \leq q-1$. The distance of these codes is at least $h+1$, since for every divisor $D \in \mathcal{D}_m$ the sous-escalier $\mathcal{N}(\text{In}(I_D))$ contains at least one of the following monomials $x^h, x^{h-1}y, \dots, y^h$, each having at least $h+1$ factors.

In fact the distance of C_m is equal to $h+1$. We obtain this equality considering the divisor D complete intersection of a vertical line \mathcal{L} of equation $x = u$, with $u \in \mathbb{F}_{q^2}$, and the curve union of $h+1$ horizontal lines passing through $h+1$ \mathbb{F}_{q^2} -points of $\mathcal{L} \cap \mathcal{H}$.

We can now describe all the divisors in \mathcal{M}_m . Since all the above monomials have more than $h+1$ factors, except x^h and y^h , for every $D \in \mathcal{M}_m$ either $y^h \in \mathcal{N}(\text{In}(I_D))$ or $x^h \in \mathcal{N}(\text{In}(I_D))$:

- If $y^h \in \mathcal{N}(\text{In}(I_D))$, then $\mathcal{N}(\text{In}(I_D))$ must be the set of divisors of y^h , namely $\text{In}(I_D) = \langle x, y^{h+1} \rangle$. Thus, we obtain precisely the divisors described above.
- If $x^h \in \mathcal{N}(\text{In}(I_D))$, then $D \in \mathcal{M}_m$ only if $m+1 \leq w(x^h) = hq$ and $\mathcal{N}(\text{In}(I_D))$ is the set of divisors of x^h , namely $\text{In}(I_D) = \langle y, x^{h+1} \rangle$. Thus, \mathcal{M}_{hq-1} also contains the divisors D complete intersection of a non-vertical line \mathcal{L}' of equation $y = ux + v$, with $(u, v) \in \mathbb{F}_{q^2}^2 \setminus \{(0, 0)\}$, and the curve union of $h+1$ vertical lines passing through $h+1$ \mathbb{F}_{q^2} -points of $\mathcal{L}' \cap \mathcal{H}$.

Note that this result is equivalent to that found in [12]. For more details see Section 6.

Example 3.1. Consider again the case $q = 4$ and the codes C_m with $m = 11, 12, 13, 14$. If D is a \mathbb{F}_{q^2} -divisor which is the support of some codewords of C_{11} , we know by Proposition 2.12 (iii) that $\mathcal{N}(\text{In}(I_D))$ contains at least one of the monomials x^3, x^2y, xy^2, y^3 , hence $|D| \geq 4$, since there are at least 4 monomials that divide each of those monomials. Then $d(m) \geq 4$.

On the other hand, by the same proposition we see that the divisor D' cut on \mathcal{H} by the line $x = 0$ is the support of a codeword of C_{14} since $\mathcal{N}(\text{In}(I_D))$ contains the monomial y^3 whose w -weight is 15, so that $d(14) \leq 4$.

Therefore, from $4 \leq d(11) \leq d(12) \leq d(13) \leq d(14) \leq 4$ we obtain that the distance of the code C_{11}, C_{12}, C_{13} , and C_{14} , is 4.

All the divisors $D \in \mathcal{M}_{14}$ are such that $\text{In}(I_D) = \langle x, y^4 \rangle$. On the converse, every divisor on \mathcal{H} of degree 4 such that $\text{In}(I_D) = \langle x, y^4 \rangle$ is in \mathcal{M}_4 if it is made of 4 different \mathbb{F}_{q^2} -points. Therefore the divisors $D \in \mathcal{M}_{14}$ are precisely the sets of 4 \mathbb{F}_{q^2} -points of

\mathcal{H} with the same x -coordinate. They are also the set of divisors in \mathcal{M}_{12} , \mathcal{M}_{13} , while in \mathcal{M}_{12} there are also those with $\text{In}(I_D) = \langle y, x^4 \rangle$.

4. THE DISTANCE OF THE HERMITIAN CODES C_m IN THE II PHASE

In this section we prove that the distance of the codes in the II phase is given by the formula $d(m) = \tilde{\delta}_m$. This result holds true also for the codes in the final part of the I phase, that we include in our statements, and for the codes of the III and IV phase already considered in [13]. Furthermore, Theorem 4.1 gives informations also about the small-weight codewords of C_m , more precisely for those of weight $d(m) + \kappa$ for some $\kappa \leq q$. Therefore we state and prove it in the widest generality, namely also including III and IV phases.

Of course, for every m , the value of $\tilde{\delta}_m$ agrees with that given by the well known formulas of the distance proved by [7, 17, 18], provided one takes in account the different notation.

We start considering the codes with $m = \tilde{m}$.

Theorem 4.1. *Let us consider an Hermitian code C_m with $m \geq q^2 - q - 1$ and $m = \tilde{m}$. Let $x^u y^v \in \mathcal{B}$ be a monomial with $w(x^u y^v) = m + 1 + \kappa$ for some $0 \leq \kappa \leq q$. If D is a divisor over \mathcal{H} such that $x^u y^v \in \mathcal{N}(\text{In}(I_D))$, then*

$$|D| \geq \tilde{\delta}_m + \kappa.$$

More precisely, if $\delta(m) = \mu q + \lambda(q + 1)$, then

- (a): if $\mu = 0$ and $0 \leq \kappa \leq q - \lambda$ then $|D| \geq \tilde{\delta}_m + \kappa(q + 1 - \lambda - \kappa)$.
- (b): if $\mu = 0$ and $\kappa \geq q - \lambda + 1$ then $|D| \geq \tilde{\delta}_m + \kappa$.
- (c): if $\mu \neq 0$ and $0 \leq \kappa \leq \mu - 1$, then $|D| \geq \tilde{\delta}_m + \kappa$.
- (d): if $\mu \neq 0$ and $\mu \leq \kappa \leq q - \lambda$, then $|D| \geq \tilde{\delta}_m + (\kappa - \mu)(q - \lambda - \kappa) + \kappa$.
- (e): if $\mu \neq 0$ and $q - \lambda + 1 \leq \kappa \leq q$, then $|D| \geq \tilde{\delta}_m + \kappa$.

Proof. First of all we observe that the minimum m that satisfies the hypotheses is $q^2 - 2$. Therefore, we may assume without lost in generality $m \geq q^2 - 2$.

We first assume that $\mu = 0$, so that $\tilde{\delta}_m = \lambda(q + 1)$. The only monomial in \mathcal{B} with w -weight $m + 1 + \kappa$ is $x^u y^v = x^{q-\kappa} y^{\lambda+\kappa-1}$.

- (a) If $q - \lambda - \kappa \geq 0$, counting the monomials that divides $x^{q-\kappa} y^{\lambda+\kappa-1}$ we get the bound:

$$|D| \geq (q - \kappa + 1)(\lambda + \kappa) = \tilde{\delta}_m + \kappa(q + 1 - \lambda - \kappa) \geq \tilde{\delta}_m + \kappa.$$

- (b) If $\lambda + \kappa - q > 0$, we observe that $\lambda + \kappa - 1 \geq q$ and $y^{\lambda+\kappa-1} \in \mathcal{N}(\text{In}(I_D))$; then we apply Lemma 2.14 and see that also the monomial $x^q y^{\lambda+\kappa-1-q}$ is

in $\mathcal{N}(\text{In}(I_D))$. Counting the monomials that divide either $x^{q-\kappa}y^{\lambda+\kappa-1}$ or $x^qy^{\lambda+\kappa-1-q}$ we get the bound

$$|D| \geq (q - \kappa + 1)(\lambda + \kappa) + \kappa(\lambda + \kappa - q) = \tilde{\delta}_m + \kappa.$$

Now suppose $\delta(m) = w(x^\mu y^\lambda)$ with $\mu > 0$. The monomial $x^u y^v$ having w -weight $m + 1 + \kappa$ is $x^{\mu-\kappa-1}y^{\lambda+q+\kappa-1}$ for $\kappa = 0, \dots, \mu - 1$ and $x^{\mu+q-\kappa}y^{\lambda+\kappa-1}$ for $\kappa = \mu, \dots, q$.

- (c) If $0 \leq \kappa \leq \mu - 1$, by Lemma 2.14 we also see that $x^qy^{\lambda+\kappa-1} \in \mathcal{N}(\text{In}_<(I_D))$. Counting the monomials that divide either $x^{\mu-\kappa-1}y^{\lambda+q+\kappa-1}$ or $x^qy^{\lambda+\kappa-1}$ we get

$$|D| \geq (\mu - \kappa)(\lambda + q + \kappa) + (q + 1 - \mu + \kappa)(\lambda + \kappa) = \tilde{\delta}_m + \kappa.$$

- (d) If $\mu \leq \kappa \leq q - \lambda$, counting the monomials that divide $x^{\mu+q-\kappa}y^{\lambda+\kappa-1}$ we get

$$|D| \geq (\mu + q - \kappa + 1)(\lambda + \kappa) = \tilde{\delta}_m + (\kappa - \mu)(q - \lambda - \kappa) + \kappa \geq \tilde{\delta}_m + \kappa.$$

- (e) If $q - \lambda + 1 \leq \kappa \leq q$, we see in the same way that $x^qy^{\lambda+\kappa-1-q} \in \mathcal{N}(\text{In}_<(I_D))$. Counting the monomials that divide either $x^{\mu+q-\kappa}y^{\lambda+\kappa-1}$ or $x^qy^{\lambda+\kappa-1-q}$ we get

$$|D| \geq (\mu + q - \kappa + 1)(\lambda + \kappa) + (q - \mu - q + \kappa)(\lambda + \kappa - q) = \tilde{\delta}_m + \kappa.$$

□

Theorem 4.2. *Let C_m be any code with $q^2 - q \leq m \leq 2q^2 - 2q - 3$. Then its distance is*

$$d(m) = \tilde{\delta}_m.$$

Moreover, there exist divisors in \mathcal{M}_m that are cut on \mathcal{H} by a suitable union of lines.

Proof. If $m = \tilde{m}$, we have already proved the result in Theorem 4.1. Then assume $\tilde{m} > m$. By Corollary 2.13 it is sufficient to prove that $d(m) \geq d(\tilde{m}) = \tilde{\delta}_m$.

By this same result it is also sufficient to prove the result for the integers m such that $\delta(m)$ is the first integer in a segment of gaps. By (2.4) and (2.5) there is an integer h such that $1 \leq h \leq q - 1$ and $m = (q + h - 3)(q + 1) + 1$, $\tilde{\delta}_m = hq = w(x^h)$.

We first assume $h \geq 2$. In this case $m = w(x^q y^{h-2}) \in \Lambda_{\mathcal{B}}$ and $\delta(m - 1) = (h - 1)(q + 1) = w(y^{h-1}) \in \Lambda$, so that the integer $m - 1$ is one of those considered in Theorem 4.1 (a) and (b). Therefore, we know that $d(m - 1) = \delta(m - 1) = (h - 1)(q + 1)$. Let D be any divisor in \mathcal{D}_m . By hypothesis, $\mathcal{N}(\text{In}(I_D))$ contains a monomial $x^u y^v$ whose w -weight is $m + 1 + t$ for some $t \geq 0$. By Corollary 2.13 D also belongs to \mathcal{D}_{m-1} and we can apply to it Theorem 4.1 (a) with $\kappa := t + 1 \geq 1$ and $\lambda = h - 1$. If $\kappa \leq q + 1 - h$, we get by (a) of Theorem 4.1,

$$|D| \geq \delta(m - 1) + \kappa(q + 1 - \lambda - \kappa) = hq + (k - 1)(q + 1 - h - k) \geq hq = \tilde{\delta}_m.$$

On the other hand, if $\kappa > q + 1 - h$ then by (b) of Theorem 4.1,

$$|D| \geq \delta(m-1) + \kappa \geq (h-1)(q+1) + (q+1-h) = hq = \tilde{\delta}_m.$$

We obtain the equality $d(m) = \tilde{\delta}_m$ proving the second assertion.

We may assume that $m = \tilde{m}$ as $\mathcal{M}_{\tilde{m}} \subseteq \mathcal{M}_m$ (Corollary 2.13). With this assumption, we can apply Theorem 4.2 and see that the equality $|D| = \tilde{\delta}_m$ can hold only when $\kappa = 0$, namely when $w(x^u y^v) = m + 1$. In particular if $\mu = 0$, the equality can hold only when $u = q$ and $v = \lambda - 1$, while if $\mu \geq 1$ the equality can hold only when $u = \mu - 1$ and $v = \lambda + q - 1$.

In both cases we obtain an \mathbb{F}_{q^2} -divisor D' of degree $\tilde{\delta}_m$ as the divisor cut on \mathcal{H} by a curve \mathcal{Y} union of lines, as we proved in Theorem 3.3 of [13]. Specifically,

- if $\mu = 0$, it is sufficient to chose \mathcal{Y} as the union of λ horizontal lines that intersect \mathcal{H} in q \mathbb{F}_{q^2} -points; the monomial $x^q y^{\lambda-1}$ with w -weight $m+1$ belongs to $\text{In}_{\prec}(I_{D'}) = \langle x^{q+1}, y^{\lambda} \rangle$, hence D' is the support of codewords of C_m by Proposition 2.12.
- if $\mu \geq 1$, it is sufficient to chose \mathcal{Y} as the union of μ vertical and λ horizontal lines, each intersecting \mathcal{H} in q and, respectively, $q+1$ \mathbb{F}_{q^2} -points; the monomial $x^{\mu-1} y^{\lambda+q-1}$ with w -weight $m+1$ belongs to $\text{In}_{\prec}(I_{D'}) = \langle x^{q+1}, x^{\mu} y^{\lambda}, y^{\lambda+q} \rangle$, hence D' is the support of codewords of C_m .

□

5. GEOMETRIC DESCRIPTION OF MINIMUM WEIGHT CODEWORDS IN THE II PHASE

In this section we focus on the \mathbb{F}_{q^2} -divisor that are the support of minimum weight codewords of codes C_m in the II phase and prove that they are all complete intersection of two curves and describe equations for each of them. In fact we prove this result assuming $q^2 - q - 1 \leq m \leq 2q^2 - 2q - 3$, a range that contains the II phase and also part of the I phase.

We first consider the case of a code C_m with $m = \tilde{m}$.

Example 5.1. Let us consider again the same case $q = 4$ of the previous examples. For $m = 18$ we have $m + 1 = 19 \in \Lambda_{\mathcal{B}}$ and $\delta(m) = 8 \in \Lambda$, so that $m = \tilde{m} = 18$. By Theorem 4.2 the distance of the C_{18} is $d(m) = \tilde{\delta}_m = \delta(m) = 8$. Let D be a divisor in \mathcal{M}_{18} . By Theorem 4.1, the only monomial in \mathcal{B} with w -degree $m + 1 = 19$, namely xy^3 , belongs to $\mathcal{N}(\text{In}(I_D))$. Then $\mathcal{N}(\text{In}(I_D)) \supseteq \{1, y, y^2, y^3, x, xy, xy^2, xy^3\}$ and the inclusion is in fact an equality, as $|\mathcal{N}(\text{In}(I_D))| = |D| = 8$. The monomial x^2 is not one of them, hence it belongs to $\text{In}(I_D)$, namely there is a curve \mathcal{X} , defined by a polynomial F with leading monomial x^2 , that contains D .

Again the inclusion $D \subseteq \mathcal{H} \cap C$ is an equality as the degree of this zero-dimensional scheme is 8 (Proposition 2.11). Therefore, D is the complete intersection of the Hermitian curve \mathcal{H} and the curve defined by a polynomial F whose leading monomial is the unique monomial in \mathcal{B} with w -weight $8 = d(m)$.

On the converse, let F be a polynomial with leading monomial x^2 , namely $F = x^2 - (sy + s'x + s'')$ that cuts on \mathcal{H} a set D of 8 points with coordinates in \mathbb{F}_{q^2} . By Proposition 2.12, $D \in \mathcal{M}_{18}$ since the the initial ideal of $I_D = \langle x^5 - y^4 - y, F \rangle = \langle y^4 + y - x(sy + s'x + s'')^2, F \rangle$ does not contain the monomial xy^3 with w -weight $19 = m + 1$.

Theorem 5.2. *Let C_m be an Hermitian with $q^2 - q - 1 \leq m \leq 2q^2 - 2q - 3$ and $m = \tilde{m}$.*

If D is a \mathbb{F}_{q^2} -divisor of degree $\delta(m) = \tilde{\delta}_m = \mu q + \lambda(q + 1)$, then $D \in \mathcal{M}_m$ if and only if $D = \mathcal{H} \cap \mathcal{X}$ with \mathcal{X} a curve defined by a polynomial F such that $\text{LM}_{\prec}(F) = x^\mu y^\lambda$.

Proof. Let D be a \mathbb{F}_{q^2} -divisor cut on \mathcal{H} by a curve \mathcal{X} defined by a polynomial F such that $\text{LM}_{\prec}(F) = x^\mu y^\lambda$. We may assume without lost of generality that F also satisfies the condition $\partial_x(F) \leq q$; in fact, if necessary, we can substitute F by its reduction modulo H , that satisfies all the required conditions $\text{LM}_{\prec}(F') = x^\mu y^\lambda$, $\partial_x(F') \leq q$ and $I_D = \langle H, F' \rangle$. Therefore we can apply Proposition 2.11 and see that $|D| = \mu + \lambda(q + 1)$ and $\mathcal{N}(\text{In}(I_D)) = \mathcal{N}(\langle x^{q+1}, x^\mu y^\lambda, y^{\lambda+q} \rangle)$ contains a monomial with w -weight larger than m ; more precisely it contains $x^{\mu-1}y^{\lambda+q-1}$ if $\mu > 0$ and $x^q y^{\lambda-1}$ if $\mu = 0$.

By Proposition 2.12 and Theorem 4.2, D belongs to \mathcal{M}_m , since $\mu q + \lambda(q + 1) = \tilde{\delta}_m = d(m)$.

On the other hand, let us assume that D is a divisor in \mathcal{M}_m . By Theorem 4.2 its degree is $\mu q + \lambda(q + 1) = \tilde{\delta}_m$; moreover, by Proposition 2.12 and Theorem 4.1, its support contains the monomial $x^a y^b \in \mathcal{B}$ such that $w(x^a y^b) = m + 1$.

We now consider the two cases related to the integer μ .

If $\mu = 0$: then $x^a y^b = x^q y^{\lambda-1}$. By Theorem 4.1 (a), the sous-escalier $\mathcal{N}(\text{In}_{\prec}(I_D))$ is the set of monomials that divides $x^q y^{\lambda-1}$. Hence y^λ belongs to $\text{In}_{\prec}(I_D)$, namely there is a polynomial F in I_D with leading monomial y^λ .

If $\mu \neq 0$: then $x^a y^b = x^{\mu-1} y^{\lambda+q-1}$. By Theorem 4.1 (c), $\mathcal{N}(\text{In}_{\prec}(I_D))$ is the set of monomials that divides either $x^{\mu-1} y^{\lambda+q-1}$ or $x^q y^{\lambda-1}$, hence there is a polynomial F in I_D with leading monomial $x^\mu y^\lambda$.

In both cases, I_D is contained in $\langle H, F \rangle$ and, by Proposition 2.11, they are in fact equal since they define zero-dimensional schemes of the same degree $d(m) = \tilde{\delta}_m$. \square

We observe that the proof of the above result is very similar to that of the analogous results for Hermitian codes of the III and IV phases in [13]. Before state and prove

the results for the other codes C_m of the II phase, we show by an example the different behavior that appears when $m < \tilde{m}$.

Example 5.3. Similarly to the previous examples, we assume $q = 4$. For $m = 16$ we have $\tilde{m} = 18$ and the distance of the code C_{16} is the same as that of C_{18} , namely $d(16) = d(18) = 8 = w(x^2)$ (Theorem 4.2 or Example 5.1).

By Theorem 5.2 the minimum codewords of C_{18} are supported on complete intersection D of \mathcal{H} and a curve \mathcal{X} defined by a polynomial F with $\text{LM}_\prec(F) = x^2$. By Proposition 2.12 we see that these divisors D belongs to \mathcal{M}_{16} .

On the other hand this same result allows the possibility that there are divisors $D \in \mathcal{M}_{16} \setminus \mathcal{M}_{18}$.

The monomials $x^u y^v \in \mathcal{B}$ with $16 + 1 \leq w(x^u y^v) < 18 + 1$ are $x^3 y$ and $x^2 y^2$. Every divisor D such that $x^2 y^2 \in \mathcal{N}(\text{In}_\prec(I_D))$ has degree at least 9, hence it does not belong to \mathcal{M}_{16} .

Then assume that $x^3 y \in \mathcal{N}(\text{In}_\prec(I_D))$, so that $I_D = (F, G)$ with $\text{LM}_\prec(F) = x^4$ and $\text{LM}_\prec(G) = y^2$. For instance we can obtain divisors of this type in the following way: every line $y - \beta$ with $\beta \in \mathbb{F}_{q^2}$, $\beta \neq 0$ intersect \mathcal{H} in 5 points (α_i, β) with $\alpha_i \in \mathbb{F}_{q^2}$. If $\beta^4 + \beta = \gamma^4 + \gamma$ then also $(\alpha_i, \gamma) \in \mathcal{H}$; the polynomials $F = \Pi(x - \alpha_i)$ and $G = (y - \beta)(y - \gamma)$ define two curves as wanted.

Note that the divisors that are support of minimum-weight codewords for C_{17} are so also for C_{16} , but the converse is not true in general. Indeed, we have $\mathcal{M}_{16} \supsetneq \mathcal{M}_{17} = \mathcal{M}_{18}$.

Theorem 5.4. *Let C_m be an Hermitian with $q^2 - q - 1 \leq m \leq 2q^2 - 2q - 3$ and $m < \tilde{m}$, so that $\tilde{\delta}_m = \mu q = w(x^\mu)$.*

A \mathbb{F}_{q^2} -divisor D over the Hermitian curve \mathcal{H} is the support of minimum weight codewords of C_m if and only if it is a complete intersection of either of the following types:

- (i) $D = \mathcal{H} \cap \mathcal{X}$ with \mathcal{X} the curve defined by a polynomial F such that $\text{LM}_\prec(F) = x^\mu$;
- (ii) $m = (\mu + q - 3)(q + 1) + 1$ with $1 \leq \mu \leq q - 1$ and $D = \mathcal{X} \cap \mathcal{Y}$ with \mathcal{X}, \mathcal{Y} curves defined by polynomials F_1 and F_2 such that $\text{LM}_\prec(F_1) = x^q$ and $\text{LM}_\prec(F_2) = y^\mu$.

Proof. By Theorem 4.2 and Corollary 2.13 we have $d(m) = d(\tilde{m}) = \mu q$ and $\mathcal{M}_{\tilde{m}} \subseteq \mathcal{M}_m$. Then, by Theorem 5.2 all the divisors in (i) are the support of codewords of C_m . Therefore, it is sufficient to prove that $\mathcal{M}_{\tilde{m}} = \mathcal{M}_m$ except for $m = (\mu + q - 3)(q + 1) + 1$ and that in this case the divisors in $\mathcal{M}_m \setminus \mathcal{M}_{\tilde{m}}$ are those described in (ii).

Let $m = (\mu + q - 3)(q + 1) + 1$ and $D \in \mathcal{M}_m \setminus \mathcal{M}_{\tilde{m}}$. By Proposition 2.12 and Theorem 5.2 we know that $\mathcal{N}(\text{In}(I_D))$ contains one of the monomials with w -weight between $m + 1$ and \tilde{m} , namely of the type $x^{q-i} y^{\mu+i-2}$ with $i = 1, \dots, q - \mu$.

Each of these monomials, except the first one, has more than μq factors, hence $D \notin \mathcal{M}_m$ if $x^{q-i} y^{\mu+i-2} \in \mathcal{N}(\text{In}(I_D))$ for some $i > 1$. The first monomial $x^{q-1} y^{\mu-1}$

has exactly μq factors and its w -weight is $m + 1$, so that the corresponding divisors do not belong to $\mathcal{M}_{m'}$ for every $m' > m$.

Moreover, we obtain a divisor D of degree μq such that $x^{q-1}y^{\mu-1} \in \mathcal{N}(\text{In}(I_D))$ only if $\mathcal{N}(\text{In}(I_D))$ is precisely the set of μq monomials that divide $x^{q-1}y^{\mu-1}$. Therefore, $x^q, y^\mu \in \text{In}(I_D)$. If F_1 and F_2 are polynomials in I_D such that $\text{LM}_<(F_1) = x^q$ and $\text{LM}_<(F_2) = y^\mu$, then they generate I_D .

Note that divisors D as above do exist; for instance we can choose q distinct elements $\alpha_i \in \mathbb{F}_{q^2}$ and μ distinct elements $\beta_j \in \mathbb{F}_{q^2}$ such that $\text{tr}(\beta_j) = N(\alpha_i)$ for every i, j and take $F_1 = \prod_{i=1}^q (x - \alpha_i)$ and $F_2 = \prod_{j=1}^\mu (y - \beta_j)$. \square

6. COMPARISON WITH THE KNOWN RESULTS

As already underlined the results we obtain about the distance of the Hermitian codes, though formulated in a different way, agree with those proved by Stichtenoth [17], Yang and Kumar [18] and Høholdt, van Lint and Pellikaan [7].

Furthermore, there is a partial overlap with known results for what concerns the geometric description of minimum weight codewords. In particular, Marcolla, Pellegrini and Sala [12] find a geometric description for the codes in the I phase, while Ballico and Ravagnani [1] find it for a few codes of the II phase. We state and prove again the known results for sake of completeness and to highlight the simplicity of our method. Moreover, the different description we propose for the divisors $D \in \mathcal{M}_m$ shows that also for the already known cases they are complete intersection.

We now compare our results with the known ones.

I phase. In [12], the authors consider Hermitian codes C_m with $d(m) \leq q$, that is, $m \leq q^2 - 2$. The result obtained in [12] is complete, but the description of minimum weight codewords is slightly different from ours. For instance in Corollary 1 of [12] the authors describe this support for some codes that they call *edge codes* in the following way “*The support of a minimum-weight codeword lies in the intersection of the Hermitian curve \mathcal{H} and a vertical line.*”

The edge codes C_m with $q^2 - q \leq m \leq q^2 - 2$ appear among those considered in Sections 4 and 5. Specializing our general results we get that the distance of such a code C_m is q and that $D \in \mathcal{M}_m$ if and only if $D \subset E$ and it is the complete intersection of \mathcal{H} and a curve \mathcal{X} defined by a polynomial F such that $LM(F) = x$; \mathcal{X} is indeed the vertical line of [12, Corollary 1] that we mention above.

A tricky case is that of the codes C_m called *corner codes* in [12], where it is proved that the set of points “*corresponding to minimum-weight codewords lies on a same line.*”

The corner code with $m = q^2 - q - 1$ is one of those considered in Theorem 5.4 (ii); in fact $m = (\mu + q - 3)(q + 1) + 1$ with $\mu = 1$. However, in our description divisors on vertical lines and divisors on non-vertical lines come from the two different families described in either (i) or (ii) of Theorem 5.4 in terms of initial ideal. Indeed, if $D \subset E$, then $D \in \mathcal{M}_{q^2-q-1}$ if and only if it is of either type :

- (i) $D = \mathcal{L} \cap \mathcal{H}$ with $\text{In}(I_D) = \langle x, y^q \rangle$ (\mathcal{L} a vertical line)
- (ii) $D = \mathcal{L}' \cap \mathcal{X}$ with $\text{In}(I_D) = \langle y, x^q \rangle$ (\mathcal{L}' a non-vertical line).

II phase. For the codes of the II phase we compare our results with [1, Theorem 19], that only concerns the codes C_m with $q^2 < m \leq q^2 + q$ corresponding to codes having distance $2q$, $2q+1$ and $2q+2$. We compare the two descriptions of the divisors $D \in \mathcal{M}_m$. In [1, Theorem 19] it is proved that

- if the distance is either $2q+2$ or $2q$, then $D \in \mathcal{M}_m$ if and only if $D \subseteq E$ and it is contained into a conic of \mathbb{P}^2 .
- if $d = 2q+1$, then $D \in \mathcal{M}_m$ if and only if $D \subset E$ and $P_\infty \cup D$ lie on a conic of \mathbb{P}^2 .

This description is substantially equivalent to ours, where the condition about P_∞ is replaced by a condition on the leading monomials of the polynomial that defines the conic.

However, in [1, Theorem 19] the case with distance $2q$ and $a = q$ (in our notation is the code C_{q^2}) is missing. Indeed, this is another tricky case of Theorem 5.4 (ii) similar to the one quoted above; in fact $m = (\mu + q - 3)(q + 1) + 1$ with $\mu = 2$. For these codes, the divisors support of minimum weight codewords are of two different types. In particular, $D \in \mathcal{M}_m$ if and only if $D \subset E$ and it is either the complete intersection of \mathcal{H} and a curve \mathcal{X} defined by a polynomial F such that $\text{LM}(F) = x^2$ or D is the complete intersection of the curves defined by two polynomials F_1 and F_2 such that $\text{LM}_<(F_1) = x^q$ and $\text{LM}_<(F_2) = y^2$.

7. CONCLUSION

With this paper the authors conclude the analysis of the supports of minimum weight codewords of the Hermitian codes, obtaining the general result (Theorem A):

For any Hermitian code C_m , the support of every minimum-weight codeword is a complete intersection of two plane curves.

In the previous paper [13] it is proved that the distance of every code C_m with $m \geq 2q^2 - 2q - 2$ (III and IV phase) is given by $d(m) = m - q^2 + q + 2$, hence there is a 1-1 correspondence between codes and distances; moreover the distance of every code can be written as $d(m) = \mu q + \lambda(q + 1)$ and a set of $d(m)$ distinct \mathbb{F}_{q^2} -points D is the support of minimum weight codewords of C_m if and only if D is cut on \mathcal{H} by a curve \mathcal{X} defined by a polynomial F with leading monomial $x^\mu y^\lambda$ w.r.t. the DegRevLex.

In the present paper the codes C_m with $m \leq 2q^2 - 2q - 3$ (I and II phase) are considered. There are a few substantial differences with respect to the codes with higher distance. The first one is that several codes may have the same distance. A second difference is that the support of minimum weight codewords of some codes C_m are complete intersections of two curves in the affine plane, but are not complete intersection in \mathcal{H} . This happens for almost all the codes in the I phase and for some special code in the II phase

- in the I phase the supports of minimum weight codewords of a code C_m are sets of $d(m)$ points on a line; when $d(m) < q$, they cannot be obtained cutting \mathcal{H} by a single curve, though they are complete intersection of a line and a curve of degree $d(m)$ (see [12, Corollary 1, Proposition 2] or Section 3).
- in the II phase, if $m \equiv 1 \pmod{q+1}$ there are two different “families” of divisors on \mathcal{H} that are support of minimum weight codewords of C_m . Indeed, there is an integer μ , $1 \leq \mu \leq q-1$, such that $m = (\mu + q - 3)(q + 1) + 1$ and the distance is $d(m) = \mu q$. The support of a minimum weight codeword can be either the divisor cut on \mathcal{H} by a curve \mathcal{X} given by a polynomial F with $\text{LM}_{\prec}(F) = x^\mu$ or the complete intersection of two curves \mathcal{X} and \mathcal{Y} defined by polynomials F_1 and F_2 such that $\text{LM}_{\prec}(F_1) = x^q$ and $\text{LM}_{\prec}(F_2) = y^\mu$ (see Theorem 5.4 (ii)).

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